JORDANIAN QUANTUM ALGEBRA $\mathcal{U}_{\mathsf{h}}(sl(N))$ VIA CONTRACTION METHOD AND MAPPING

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Abstract

Using the contraction procedure introduced by us in Ref. [20], we construct, in the first part of the present letter, the Jordanian quantum Hopf algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ which has a remarkably simple coalgebraic structure and contains the Jordanian Hopf algebra $\mathcal{U}_{\mathsf{h}}(sl(2))$, obtained by Ohn, as a subalgebra. A nonlinear map between $\mathcal{U}_{\mathsf{h}}(sl(3))$ and the classical $sl(3)$ algebra is then established. In the second part, we give the higher dimensional Jordanian algebras $\mathcal{U}_{\mathsf{h}}(sl(N))$ for all N. The Universal $\mathcal{R}_{\mathsf{h}}$-matrix of $\mathcal{U}_{\mathsf{h}}(sl(N))$ is also given.

Keywords: Standard quantization, Nonstandard quantization, contraction procedure, Hopf algebra, universal $\mathcal{R}$-matrix, Irreducible representations (irreps.).

1 Introduction

It is well known that the enveloping Lie algebra $\mathcal{U}(sl(N))$ has two quantizations: The first one called the *Drinfeld-Jimbo deformation* or the *standard quantum deformation* [1, 2] is quasitriangular ($\mathcal{R}_{21}\mathcal{R} \neq I$), whereas the second one called the *Jordanian deformation* or the *non-standard quantum deformation* [3] is triangular ($\mathcal{R}_{21}\mathcal{R} = I$). A typical example of Jordanian quantum algebras was first introduced by Ohn [4]. In general, nonstandard quantum algebras are obtained by applying Drinfeld twist to the corresponding Lie algebras [5]. The twisting that produces an algebra isomorphic to the Ohn algebra [4] is found in [6, 7].

Recently, the twisting procedure was extensively employed to study a wide variety of Jordanian deformed algebras, such as $\mathcal{U}_{\mathsf{h}}(sl(N))$ algebras [8, 9, 10, 11], symplectic algebras

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$\mathcal{U}_{\mathsf{h}}(sp(N))$ [12], orthogonal algebras $\mathcal{U}_{\mathsf{h}}(so(N))$ [13, 14, 15, 16] and orthosymplectic superalgebra $\mathcal{U}_{\mathsf{h}}(osp(1|2))$ [17, 18]. It appears from these studies that:

1. The non-standard quantum algebras have undeformed commutation relations;

2. The Jordanian deformation appear only in the coalgebraic structure;

3. The coproduct and the antipode maps have very complicated forms in comparison with the Drinfeld-Jimbo and the Ohn deformations.

To our knowledge, Jordanian quantum algebra $\mathcal{U}_{\mathsf{h}}(sl(N))$ has been written explicitly, with a simple coalgebra, only for $N = 2$ [4]. The main object of the present letter is to construct the Jordanian quantum algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ using the contraction procedure developed in [20] and the map studied in Refs. [20, 21]. The $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra presented here has the following properties:

1. The Ohn algebra $\mathcal{U}_{\mathsf{h}}(sl(2))$ is included in our structure $\mathcal{U}_{\mathsf{h}}(sl(3))$ in a natural way as a Hopf subalgebra and appear here from the longest root generators *i.e.* from e_3, f_3 and their corresponding Cartan generator h_3;

2. Our Jordanian deformed $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra may be regarded as the dual Hopf algebra of the function algebra $Fun_{\mathsf{h}}(SL(3))$ studied in [22];

3. The present $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra is endowed with a relatively simple coalgebra structure (as compared to previous studies [8, 9, 10, 11]).

Implementing our contraction technique we subsequently obtain higher dimensional Jordanian quantum algebras $\mathcal{U}_{\mathsf{h}}(sl(N))$ for arbitrary values of N.

This letter is organized as follows: The Jordanian quantum algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ is introduced via a nonlinear map and proved to be a Hopf algebra in section 2. The irreducible representations (irreps.) of $\mathcal{U}_{\mathsf{h}}(sl(3))$ are also given. Higher dimensional algebras $\mathcal{U}_{\mathsf{h}}(sl(N))$, $N \geq 4$ are presented in the sections 3 and 4.

2 $\mathcal{U}_{\mathsf{h}}(sl(3))$: Map, Hopf Algebra, Irreps. and $\mathcal{R}_{\mathsf{h}}$-matrix

In this letter, h is an arbitrary complex number. It was proved in [20] that the $\mathcal{R}_{\mathsf{h}}$-matrix of the Jordanian quantum algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ can be obtained from the $\mathcal{R}_q$-matrix associated to the Drinfeld-Jimbo quantum algebra $\mathcal{U}_q(sl(3))$ through a specific contraction which is singular in the $q \to 1$ limit. For the transformed matrix, the singularities, however, cancel yielding a well-defined construction. Here we assume the $\mathcal{U}_q(sl(3))$ Hopf algebra to be well-known [23].

For brevity and simplicity we limit ourselves to (fundamental irrep.) $\otimes$ (arbitrary irrep.). Recall that for $\mathcal{U}_q(sl(3))$ algebra the R_q-matrix in the representation (fund.) $\otimes$ (arb.) reads [23]:

$$\begin{aligned} R_q &= \Big(\pi_{(fund.)} \otimes \pi_{(arb.)}\Big)\mathcal{R}_q \\ &= \begin{pmatrix} q^{\frac{1}{3}(2h_1+h_2)} & q^{\frac{1}{3}(2h_1+h_2)}\Lambda_{12} & q^{\frac{1}{3}(2h_1+h_2)}\Lambda_{13} \\ 0 & q^{-\frac{1}{3}(h_1-h_2)} & q^{-\frac{1}{3}(h_1-h_2)}\Lambda_{23} \\ 0 & 0 & q^{-\frac{1}{3}(h_1+2h_2)} \end{pmatrix}, \end{aligned} \tag{1}$$

where

$$
\begin{aligned}
\Lambda_{12} &= q^{-1/2}(q-q^{-1})q^{-h_1/2}\hat{f}_1, \\
\Lambda_{13} &= q^{-1/2}(q-q^{-1})\hat{f}_3 q^{-\frac{1}{2}(h_1+h_2)}, \\
\Lambda_{23} &= q^{-1/2}(q-q^{-1})q^{-h_2/2}\hat{f}_2.
\end{aligned} \tag{2}
$$

The elements $k_1^{\pm 1} = q^{\pm h_1}$, $k_2^{\pm 1} = q^{\pm h_2}$, $k_3^{\pm 1} = q^{\pm h_3} = q^{\pm(h_1+h_2)}$, $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3 = \hat{e}_1\hat{e}_2 - q^{-1}\hat{e}_2\hat{e}_1$, $\hat{f}_1$, $\hat{f}_2$ and $\hat{f}_3 = \hat{f}_2\hat{f}_1 - q\hat{f}_1\hat{f}_2$ are the $\mathcal{U}_q(sl(3))$ generators. The corresponding classical generators are denoted by h_1, h_2, $h_3 = h_1 + h_2$, e_1, e_2, $e_3 = e_1e_2 - e_2e_1$, f_1, f_2 and $f_3 = f_2f_1 - f_1f_2$.

We have shown in [20] that the nonstandard R_{h}-matrix (in the representation (fund.) $\otimes$ (arb.)) arise from the R_q-matrix (in (fund.) $\otimes$ (arb.)) as follows:

$$
\begin{aligned}
R_{\mathsf{h}} &= \lim_{q\to 1}\Big[E_q\Big(\frac{\mathsf{h}\hat{e}_3}{q-1}\Big)_{(fund.)} \otimes E_q\Big(\frac{\mathsf{h}\hat{e}_3}{q-1}\Big)_{(arb.)}\Big]^{-1} R_q \Big[E_q\Big(\frac{\mathsf{h}\hat{e}_3}{q-1}\Big)_{(fund.)} \otimes E_q\Big(\frac{\mathsf{h}\hat{e}_3}{q-1}\Big)_{(arb.)}\Big] \\
&= \lim_{q\to 1}\begin{pmatrix} E_q^{-1}(\frac{\mathsf{h}\hat{e}_3}{q-1}) & 0 & -\frac{\mathsf{h}}{q-1}E_q^{-1}(\frac{\mathsf{h}\hat{e}_3}{q-1}) \\ 0 & E_q^{-1}(\frac{\mathsf{h}\hat{e}_3}{q-1}) & 0 \\ 0 & 0 & E_q^{-1}(\frac{\mathsf{h}\hat{e}_3}{q-1}) \end{pmatrix} R_q \begin{pmatrix} E_q(\frac{\mathsf{h}\hat{e}_3}{q-1}) & 0 & \frac{\mathsf{h}}{q-1}E_q(\frac{\mathsf{h}\hat{e}_3}{q-1}) \\ 0 & E_q(\frac{\mathsf{h}\hat{e}_3}{q-1}) & 0 \\ 0 & 0 & E_q(\frac{\mathsf{h}\hat{e}_3}{q-1}) \end{pmatrix} \\
&= \begin{pmatrix} T & 2\mathsf{h}T^{-1/2}e_2 & -\frac{\mathsf{h}}{2}(T+T^{-1})(h_1+h_2) + \frac{\mathsf{h}}{2}(T-T^{-1}) \\ 0 & I & -2\mathsf{h}T^{1/2}e_1 \\ 0 & 0 & T^{-1} \end{pmatrix},
\end{aligned} \tag{3}
$$

where

$$
T = \mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2}, \qquad T^{-1} = -\mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2}. \tag{4}
$$

The deformed exponential in (3) is defined by

$$
\begin{aligned}
E_q(x) &= \sum_{n=0}^{\infty} \frac{x^n}{[n]!}, \\
[n] &= \frac{q^n - q^{-n}}{q-q^{-1}}, \qquad [n]! = [n]\times[n-1]!, \qquad [0]! = 1.
\end{aligned} \tag{5}
$$

The following properties can be pointed out:

1. The corner elements of (3) have exactly the same structure as in the R_{h}-matrix of $\mathcal{U}_{\mathsf{h}}(sl(2))$. This implies that the classical generators e_3, $h_3 = h_1 + h_2$ and f_3 of $\mathcal{U}(sl(3))$ are deformed (for the nonstandard quantization: $\mathcal{U}(sl(3)) \longrightarrow \mathcal{U}_{\mathsf{h}}(sl(3))$) as follows [20, 21]:

$$
\begin{aligned}
T &= \mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2}, \qquad & T^{-1} &= -\mathsf{h}e_3 + \sqrt{1+\mathsf{h}^2e_3^2}, \\
H_3 &= \sqrt{1+\mathsf{h}^2e_3^2}\,h_3, \qquad & F_3 &= f_3 - \frac{\mathsf{h}^2}{4}e_3(h_3^2 - 1),
\end{aligned} \tag{6}
$$

and evidently satisfy the commutation relations [4]

$$
\begin{aligned}
&TT^{-1} = T^{-1}T = 1, \\
&[H_3, T] = T^2 - 1, \qquad\qquad [H_3, T^{-1}] = T^{-2} - 1, \\
&[T, F_3] = \frac{\mathsf{h}}{2}\Big(H_3 T + T H_3\Big), \qquad [T^{-1}, F_3] = -\frac{\mathsf{h}}{2}\Big(H_3 T^{-1} + T^{-1} H_3\Big), \\
&[H_3, F_3] = -\frac{1}{2}\Big(T F_3 + F_3 T + T^{-1} F_3 + F_3 T^{-1}\Big).
\end{aligned} \tag{7}
$$

With the following definition (see Ref. [4])

$$
E_3 = \mathsf{h}^{-1} \ln T = \mathsf{h}^{-1} \operatorname{arcsinh} \mathsf{h} e_3, \tag{8}
$$

it follows that the elements H_3, E_3 and F_3 satisfy the relations

$$
\begin{aligned}
&[H_3, E_3] = 2\frac{\sinh \mathsf{h} E_3}{\mathsf{h}}, \\
&[H_3, F_3] = -F_3\Big(\cosh \mathsf{h} E_3\Big) - \Big(\cosh \mathsf{h} E_3\Big) F_3, \\
&[E_3, F_3] = H_3,
\end{aligned} \tag{9}
$$

where it is obvious that as $\mathsf{h} \longrightarrow 0$, we have $(H_3, E_3, F_3) \longrightarrow (h_3, e_3, f_3)$. It is now evident from (7) that $\mathcal{U}_{\mathsf{h}}(sl(2)) \subset \mathcal{U}_{\mathsf{h}}(sl(3))$.

2. The expression (3) of the R_{h}-matrix indicates that the simple root generators e_1 and e_2 are deformed as follows:

$$
\begin{aligned}
E_1 &= \sqrt{\mathsf{h} e_3 + \sqrt{1 + \mathsf{h}^2 e_3^2}}\, e_1 = T^{1/2} e_1, \\
E_2 &= \sqrt{\mathsf{h} e_3 + \sqrt{1 + \mathsf{h}^2 e_3^2}}\, e_2 = T^{1/2} e_2.
\end{aligned} \tag{10}
$$

To complete our $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra, we introduce the following h-deformed generators:

$$
\begin{aligned}
F_1 &= \sqrt{-\mathsf{h} e_3 + \sqrt{1 + \mathsf{h}^2 e_3^2}}\, f_1 + \frac{\mathsf{h}}{2}\sqrt{\mathsf{h} e_3 + \sqrt{1 + \mathsf{h}^2 e_3^2}}\, e_2 h_3 = T^{-1/2}\Big(f_1 + \frac{\mathsf{h}}{2} e_2 T h_3\Big), \\
F_2 &= \sqrt{-\mathsf{h} e_3 + \sqrt{1 + \mathsf{h}^2 e_3^2}}\, f_2 - \frac{\mathsf{h}}{2}\sqrt{\mathsf{h} e_3 + \sqrt{1 + \mathsf{h}^2 e_3^2}}\, e_1 h_3 = T^{-1/2}\Big(f_2 - \frac{\mathsf{h}}{2} e_1 T h_3\Big), \\
H_1 &= \Big(-\mathsf{h} e_3 + \sqrt{1 + \mathsf{h}^2 e_3^2}\Big)\Big(\sqrt{1 + \mathsf{h}^2 e_3^2}\, h_1 + \frac{\mathsf{h}}{2} e_3 (h_1 - h_2)\Big) = h_1 - \frac{\mathsf{h}}{2} e_3 T^{-1} h_3, \\
H_2 &= \Big(-\mathsf{h} e_3 + \sqrt{1 + \mathsf{h}^2 e_3^2}\Big)\Big(\sqrt{1 + \mathsf{h}^2 e_3^2}\, h_2 - \frac{\mathsf{h}}{2} e_3 (h_1 - h_2)\Big) = h_2 - \frac{\mathsf{h}}{2} e_3 T^{-1} h_3.
\end{aligned} \tag{11}
$$

The expressions (6), (10) and (11) constitute a realization of the Jordanian algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ with the classical generators via a nonlinear map. This immediately yields the irreducible representations (irreps.) of $\mathcal{U}_{\mathsf{h}}(sl(3))$ in an explicit and simple manner.

Proposition 1 *The Jordanian algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ is an associative algebra over $\mathbb{C}$ generated by H_1, H_2, H_3, E_1, E_2, T, T^{-1}, F_1, F_2 and F_3, satisfying, along with (7), the commutation relations*

$$
\begin{aligned}
&[H_1,H_2]=0, && [H_1,T^{-1}H_3]=[H_2,T^{-1}H_3]=0,\\
&[H_1,E_1]=2E_1, && [H_2,E_2]=2E_2,\\
&[H_1,E_2]=-E_2, && [H_2,E_1]=-E_1,\\
&[T^{-1}H_3,E_1]=E_1, && [T^{-1}H_3,E_2]=E_2,\\
&[H_1,F_1]=-2F_1+\mathsf{h}E_2T^{-1}H_3, && [H_2,F_2]=-2F_2-\mathsf{h}E_1T^{-1}H_3,\\
&[H_1,F_2]=F_2-\mathsf{h}E_1T^{-1}H_3, && [H_2,F_1]=F_1+\mathsf{h}E_2T^{-1}H_3,\\
&[TH_3,F_1]=-T^2F_1, && [TH_3,F_2]=-T^2F_2,\\
&[T^{-1}E_1,F_1]=\frac{1}{2}(T+T^{-1})H_1+\frac{1}{2}(T-T^{-1})H_2, &&\\
&[T^{-1}E_2,F_2]=\frac{1}{2}(T+T^{-1})H_2+\frac{1}{2}(T-T^{-1})H_1, &&\\
&[T^{-1}E_1,F_2]=0, && [T^{-1}E_2,F_1]=0,\\
&[E_1,E_2]=\frac{1}{2\mathsf{h}}(T^2-1), &&\\
&[TF_2,TF_1]=T\Big(F_3-\frac{\mathsf{h}}{2}H_3TH_3-\frac{\mathsf{h}}{8}(T-T^{-1})\Big) &&\\
&[TH_1,T]=\frac{1}{2}(T^2-1), && [TH_1,T^{-1}]=\frac{1}{2}(T^{-2}-1),\\
&[TH_2,T]=\frac{1}{2}(T^2-1), && [TH_2,T^{-1}]=\frac{1}{2}(T^{-2}-1),\\
&[H_1,F_3]=-\frac{T^{-1}}{4}\Big(TF_3+F_3T+T^{-1}F_3+F_3T^{-1}\Big)-\frac{\mathsf{h}}{4}T^{-1}H_3^2-\frac{\mathsf{h}}{4}H_3T^{-1}H_3, &&\\
&[H_2,F_3]=-\frac{T^{-1}}{4}\Big(TF_3+F_3T+T^{-1}F_3+F_3T^{-1}\Big)-\frac{\mathsf{h}}{4}T^{-1}H_3^2-\frac{\mathsf{h}}{4}H_3T^{-1}H_3, &&\\
&[E_1,T]=[E_1,T^{-1}]=[E_2,T]=[E_2,T^{-1}]=0, &&\\
&[F_1,T]=\mathsf{h}TE_2, && [F_1,T^{-1}]=-\mathsf{h}T^{-1}E_2,\\
&[F_2,T]=-\mathsf{h}TE_1, && [F_2,T^{-1}]=\mathsf{h}T^{-1}E_1,\\
&[E_1,F_3]=-\frac{1}{2}\Big(TF_2+F_2T\Big), && [E_2,F_3]=\frac{1}{2}\Big(TF_1+F_1T\Big),\\
&[F_1,F_3]=\mathsf{h}TF_1-\mathsf{h}E_2F_3+\frac{\mathsf{h}^2}{4}TE_2, &&\\
&[F_2,F_3]=\mathsf{h}TF_2+\mathsf{h}E_1F_3-\frac{\mathsf{h}^2}{4}TE_1. &&
\end{aligned}
\tag{12}
$$

Here we quoted only the final results. To obtain the realizations of H_1 and H_2 given in (11), we, in analogy with (6), started with the ansatz $\sqrt{1+\mathsf{h}^2e_3^2}h_1$ and $\sqrt{1+\mathsf{h}^2e_3^2}h_2$ for these

generators respectively. It is easy to see that

$$
\begin{aligned}
[\sqrt{1+\mathsf{h}^2e_3^2}h_1, F_3] &= -\frac{1}{4}\Big(TF_3+F_3T+T^{-1}F_3+F_3T^{-1}\Big)\\
&\quad+\frac{\mathsf{h}^2}{4}\Big(e_3(h_1-h_2)H_3+H_3e_3(h_1-h_2)\Big),\\
[\sqrt{1+\mathsf{h}^2e_3^2}h_2, F_3] &= -\frac{1}{4}\Big(TF_3+F_3T+T^{-1}F_3+F_3T^{-1}\Big)\\
&\quad-\frac{\mathsf{h}^2}{4}\Big(e_3(h_1-h_2)H_3+H_3e_3(h_1-h_2)\Big).
\end{aligned}
\tag{13}
$$

Then, if we add to $\sqrt{1+\mathsf{h}^2e_3^2}h_1$ and deduct from $\sqrt{1+\mathsf{h}^2e_3^2}h_2$ the term $\frac{\mathsf{h}}{2}e_3(h_1-h_2)$, we obtain

$$
\begin{aligned}
[(\sqrt{1+\mathsf{h}^2e_3^2}h_1+\frac{\mathsf{h}}{2}e_3(h_1-h_2)), F_3] &= -\frac{1}{4}\Big(TF_3+F_3T+T^{-1}F_3+F_3T^{-1}\Big)\\
&\quad+\frac{\mathsf{h}}{4}T(h_1-h_2)H_3+\frac{\mathsf{h}}{4}H_3T(h_1-h_2),\\
[(\sqrt{1+\mathsf{h}^2e_3^2}h_2-\frac{\mathsf{h}}{2}e_3(h_1-h_2)), F_3] &= -\frac{1}{4}\Big(TF_3+F_3T+T^{-1}F_3+F_3T^{-1}\Big)\\
&\quad-\frac{\mathsf{h}}{4}T(h_1-h_2)H_3-\frac{\mathsf{h}}{4}H_3T(h_1-h_2).
\end{aligned}
\tag{14}
$$

These commutation relations suggest the realizations $H_1 \sim \Big(\sqrt{1+\mathsf{h}^2e_3^2}h_1+\frac{\mathsf{h}}{2}e_3(h_1-h_2)\Big)$ and $H_2 \sim \Big(\sqrt{1+\mathsf{h}^2e_3^2}h_2-\frac{\mathsf{h}}{2}e_3(h_1-h_2)\Big)$. Finally, to preserve the Cartan subalgebra, we are obliged to multiply both of these expressions by T^{-1}. The resultant maps for H_1 and H_2 are quoted in (11). The expressions of F_1 and F_2 are obtained in a similar way. The expressions (6), (10) and (11) may be looked now as a particular realization of the $\mathcal{U}_{\mathsf{h}}(sl(3))$ generators. Other maps may also be considered.

Proposition 2 *In terms of the Chevalley generators (simple roots) $\{E_1, E_2, F_1, F_2, H_1, H_2\}$, the algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ is defined as follows:*

$$
\begin{aligned}
&T=\Big(1+2\mathsf{h}[E_1,E_2]\Big)^{1/2}, && T^{-1}=\Big(1+2\mathsf{h}[E_1,E_2]\Big)^{-1/2},\\
&[H_1,H_2]=0, &&\\
&[H_1,E_1]=2E_1, && [H_2,E_2]=2E_2,\\
&[H_1,E_2]=-E_2, && [H_2,E_1]=-E_1,\\
&[H_1,F_1]=-2F_1+\mathsf{h}E_2(H_1+H_2), && [H_2,F_2]=-2F_2-\mathsf{h}E_1(H_1+H_2),\\
&[H_1,F_2]=F_2-\mathsf{h}E_1(H_1+H_2), && [H_2,F_1]=F_1+\mathsf{h}E_2(H_1+H_2),\\
&[T^{-1}E_1,F_1]=\frac{1}{2}(T+T^{-1})H_1+\frac{1}{2}(T-T^{-1})H_2, &&
\end{aligned}
$$

$$
\begin{aligned}
&[T^{-1}E_2, F_2] = \frac{1}{2}(T+T^{-1})H_2 + \frac{1}{2}(T-T^{-1})H_1,\\
&[T^{-1}E_1, F_2] = [T^{-1}E_2, F_1] = 0,\\
&E_1^2E_2 - 2E_1E_2E_1 + E_2E_1^2 = 0,\\
&E_2^2E_1 - 2E_2E_1E_2 + E_1E_2^2 = 0,\\
&(TF_1)^2TF_2 - 2TF_1TF_2TF_1 + TF_2(TF_1)^2 = 0,\\
&(TF_2)^2TF_1 - 2TF_2TF_1TF_2 + TF_1(TF_2)^2 = 0,
\end{aligned} \tag{15}
$$

or, briefly

$$
\begin{aligned}
&[H_i, H_j] = 0,\\
&[H_i, E_j] = a_{ij}E_j, \qquad\qquad [H_i, F_j] = -a_{ij}F_j + T^{-1}[F_j, T](H_1+H_2),\\
&[T^{-1}E_i, F_j] = \delta_{ij}\Big(T^{-1}H_i + \frac{1}{2}(T-T^{-1})(H_1+H_2)\Big),\\
&(ad\,E_i)^{1-a_{ij}}(E_j) = 0, \qquad i\neq j,\\
&(ad\,TF_i)^{1-a_{ij}}(TF_j) = 0, \qquad i\neq j,
\end{aligned} \tag{16}
$$

where $(a_{ij})_{i,j=1,2}$ is the Cartan matrix of $sl(3)$, i.e. $a_{11}=a_{22}=2$ and $a_{12}=a_{21}=-1$.

3. We now turn to the coalgebraic structure:

Proposition 3 *The Jordanian quantum algebra $\mathcal{U}_{\mathsf{h}}(sl(3))$ admits a Hopf structure with coproducts, antipodes and counits determined by*

$$
\begin{aligned}
&\Delta(E_1) = E_1\otimes 1 + T\otimes E_1,\\
&\Delta(E_2) = E_2\otimes 1 + T\otimes E_2,\\
&\Delta(T) = T\otimes T, \qquad\qquad \Delta(T^{-1}) = T^{-1}\otimes T^{-1},\\
&\Delta(F_1) = F_1\otimes 1 + T^{-1}\otimes F_1 + \mathsf{h}H_3\otimes E_2\\
&\qquad\quad = F_1\otimes 1 + T^{-1}\otimes F_1 + T(H_1+H_2)\otimes T^{-1}[F_1, T],\\
&\Delta(F_2) = F_2\otimes 1 + T^{-1}\otimes F_2 - \mathsf{h}H_3\otimes E_1\\
&\qquad\quad = F_2\otimes 1 + T^{-1}\otimes F_2 + T(H_1+H_2)\otimes T^{-1}[F_2, T],\\
&\Delta(F_3) = F_3\otimes T + T^{-1}\otimes F_3,\\
&\Delta(H_1) = H_1\otimes 1 + 1\otimes H_1 - \frac{1}{2}(1-T^{-2})\otimes T^{-1}H_3\\
&\qquad\quad = H_1\otimes 1 + 1\otimes H_1 - \frac{1}{2}(1-T^{-2})\otimes (H_1+H_2),\\
&\Delta(H_2) = H_2\otimes 1 + 1\otimes H_2 - \frac{1}{2}(1-T^{-2})\otimes T^{-1}H_3\\
&\qquad\quad = H_2\otimes 1 + 1\otimes H_2 - \frac{1}{2}(1-T^{-2})\otimes (H_1+H_2),\\
&\Delta(H_3) = H_3\otimes T + T^{-1}\otimes H_3,
\end{aligned}
$$

$$
\begin{aligned}
&S(E_1) = -T^{-1}E_1, && S(E_2) = -T^{-1}E_2,\\
&S(T) = T^{-1}, && S(T^{-1}) = T,\\
&S(F_1) = -TF_1 + \mathsf{h}TH_3T^{-1}E_2 = -TF_1 + T^2(H_1+H_2)T^{-2}[F_1,T], &&\\
&S(F_2) = -TF_2 - \mathsf{h}TH_3T^{-1}E_1 = -TF_2 + T^2(H_1+H_2)T^{-2}[F_2,T], &&\\
&S(F_3) = -TF_3T^{-1}, &&\\
&S(H_1) = -H_1 - \frac{1}{2}(T-T^{-1})H_3 = -H_1 - \frac{1}{2}(T^2-1)(H_1+H_2), &&\\
&S(H_2) = -H_2 - \frac{1}{2}(T-T^{-1})H_3 = -H_2 - \frac{1}{2}(T^2-1)(H_1+H_2), &&\\
&S(H_3) = -TH_3T^{-1}, &&\\
&\epsilon(a) = 0, \qquad \forall a \in \Big\{H_1, H_2, H_3, E_1, E_2, F_1, F_2, F_3\Big\}, &&\\
&\epsilon(T) = \epsilon(T^{-1}) = 1. &&
\end{aligned}
\tag{17}
$$

All the Hopf algebra axioms can be verified by direct calculations. Let us remark that our coproducts have simpler forms as compared to Refs. [8, 9, 10, 11].

Proposition 4 *The universal $\mathcal{R}_{\mathsf{h}}$-matrix has the following form:*

$$\mathcal{R}_{\mathsf{h}} = \mathcal{F}_{21}^{-1}\mathcal{F}, \tag{18}$$

where

$$\mathcal{F} = \exp\Big(\mathsf{h}TH_3 \otimes E_3\Big)\exp\Big(2\mathsf{h}TE_1 \otimes T^{-2}E_2\Big). \tag{19}$$

The $\mathcal{R}$-matrix properties are verified using MAPLE. The element (18) coincides with the universal $\mathcal{R}$-matrix of the Borel subalgebra and gives exactly the expression (3) in the representation (fund.) $\otimes$ (arb.).

4. Following Drinfeld's arguments [5], it is possible to construct a twist operator $G \in \mathcal{U}(sl(3))^{\otimes 2}[[\mathsf{h}]]$ relating the Jordanian coalgebraic structure given by (17) with the corresponding classical coalgebraic structure. For an invertible map $m : \mathcal{U}_{\mathsf{h}}(sl(3)) \to \mathcal{U}(sl(3))$, $m^{-1} : \mathcal{U}(sl(3)) \to \mathcal{U}_{\mathsf{h}}(sl(3))$, the following relations hold:

$$
\begin{aligned}
(m\otimes m)\circ\Delta\circ m^{-1}(\mathcal{X}) &= G\Delta_0(\mathcal{X})G^{-1},\\
m\circ S\circ m^{-1}(\mathcal{X}) &= gS_0(\mathcal{X})g^{-1},
\end{aligned}
\tag{20}
$$

where $\mathcal{X} \in \mathcal{U}(sl(3))[[\mathsf{h}]]$ and $(\Delta_0, \epsilon_0, S_0)$ are the coproduct, counit and the antipode maps of the classical $\mathcal{U}(sl(3))$ algebra. The transforming operator $g(\in \mathcal{U}(sl(3))[[\mathsf{h}]])$ and its inverse may be expressed as

$$g = \mu\circ(\mathrm{id}\otimes S_0)G, \qquad g^{-1} = \mu\circ(S_0\otimes\mathrm{id})G^{-1}, \tag{21}$$

where μ is the multiplication map.

For the map presented here in (6), (10) and (11), we have the construction

$$
\begin{aligned}
G = 1 \otimes 1 &- \frac{1}{2}\mathsf{h}\hat{r} + \frac{1}{8}\mathsf{h}^2\Big[\hat{r}^2 + 2(e_3 \otimes e_3)\Delta_0(h_3)\Big] \\
&- \frac{1}{48}\mathsf{h}^3\Big[\hat{r}^3 + 6(e_3 \otimes e_3)\Delta_0(h_3)\hat{r} - 4(\Delta_0(e_3))^2\hat{r}\Big] \\
&+ \frac{1}{384}\mathsf{h}^4\Big[\hat{r}^4 - 16(\Delta_0(e_3))^2\hat{r}^2 + 12(e_3 \otimes e_3)\Delta_0(h_3)\hat{r}^2 + 12((e_3 \otimes e_3)\Delta_0(h_3))^2 \\
&+ 6({e_3}^2 \otimes 1 - 1 \otimes {e_3}^2)^2\Delta_0(h_3) + 12(\Delta_0(e_3))^2({e_3}^2 \otimes 1 + 1 \otimes {e_3}^2)\Delta_0(h_3) \\
&- 8\Delta_0(e_3)({e_3}^3 \otimes 1 + 1 \otimes {e_3}^3)\Delta_0(h_3) - 10(\Delta_0(e_3))^4\Delta_0(h_3)\Big] + O(\mathsf{h}^5), \\
g = 1 &+ \mathsf{h}e_3(1 + \mathsf{h}^2{e_3}^2)^{1/2} + \mathsf{h}^2{e_3}^2,
\end{aligned} \tag{22}
$$

where $\hat{r} = h_3 \otimes e_3 - e_3 \otimes h_3$. The above twist operators, while obeying the requirement (20) for the full $\mathcal{U}(sl(3))[[\mathsf{h}]]$ algebra, are, however, generated only by the elements (e_3, h_3), related to the longest root. This property accounts for the embedding of the $\mathcal{U}_{\mathsf{h}}(sl(2))$ algebra in the higher dimensional $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra. The transforming operator g is obtained in (22) in a closed form. The series expansion of the twist operator G may be developed upto an arbitrary order in h. The expansion (22) of the twist operator G in powers of h satisfies the cocycle condition

$$(1 \otimes G)(\mathrm{id} \otimes \Delta_0)G = (G \otimes 1)(\Delta_0 \otimes \mathrm{id})G \tag{23}$$

upto the desired order. The present discussion of the twist operator relating to the $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra may be easily extended to higher dimensional Jordanian algebras. (A systematic study of twists for $\mathcal{U}_{\mathsf{h}}(sl(2))$ can be found in [21]).

5. Let us mention that there is a $\mathbb{C}$-algebra automorphism ϕ of $\mathcal{U}_{\mathsf{h}}(sl(3))$ such that

$$
\begin{array}{lll}
\phi(T^{\pm 1}) = T^{\pm 1}, & \phi(F_3) = F_3, & \phi(H_3) = H_3, \\
\phi(E_1) = E_2, & \phi(F_1) = F_2, & \phi(H_1) = H_2, \\
\phi(E_2) = -E_1, & \phi(F_2) = -F_1, & \phi(H_2) = H_1.
\end{array} \tag{24}
$$

(For $\mathsf{h} = 0$, this automorphism reduces to the classical one $(h_1, e_1, f_1, h_2, e_2, f_2) \longrightarrow (h_2, e_2, f_2, h_1, -e_1, -f_1)$). Also there is a second $\mathbb{C}$-algebra automorphism φ of $\mathcal{U}_{\mathsf{h}}(sl(3))$ defined as:

$$
\begin{array}{lll}
\varphi(T^{\pm 1}) = -T^{\pm 1}, & \varphi(F_3) = -F_3, & \varphi(H_3) = -H_3, \\
\varphi(E_1) = E_1, & \varphi(F_1) = F_1, & \varphi(H_1) = H_1, \\
\varphi(E_2) = E_2, & \varphi(F_2) = F_2, & \varphi(H_2) = H_2.
\end{array} \tag{25}
$$

6. The expressions (6), (10) and (11) permit immediate explicit construction of the finite-dimensional irreducible representations of $\mathcal{U}_{\mathsf{h}}(sl(3))$. For example, the three-dimensional irreducible representations are spanned by

$$
H_1 = \begin{pmatrix} 1 & 0 & \frac{\mathsf{h}}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -\frac{\mathsf{h}}{2} \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
H_2 = \begin{pmatrix} 0 & 0 & \frac{\mathsf{h}}{2} \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 & -\frac{\mathsf{h}}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
$$

$$
H_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad T^{\pm 1} = \begin{pmatrix} 1 & 0 & \pm\mathsf{h} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{26}
$$

or, by

$$
H_1 = \begin{pmatrix} 1 & 0 & \frac{\mathsf{h}}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -\frac{\mathsf{h}}{2} \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
H_2 = \begin{pmatrix} 0 & 0 & \frac{\mathsf{h}}{2} \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 & -\frac{\mathsf{h}}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
$$

$$
H_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T^{\pm 1} = \begin{pmatrix} -1 & 0 & \mp\mathsf{h} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{27}
$$

The three-irrep. (27) is simply obtained form the irrep. (26) using the automorphism φ. The irrep. (27) has evidently no classical ($\mathsf{h} = 0$) limit.

3 $\mathcal{U}_{\mathsf{h}}(sl(4))$: Map and $\mathcal{R}_{\mathsf{h}}$-matrix

The major interest of our approch is that it can be generalized for obtaining Jordanian quantum algebras $\mathcal{U}_{\mathsf{h}}(sl(N))$ of higher dimensions. Here we illustrate our method using $\mathcal{U}(sl(4))$ as an example. Let $h_1 = e_{11} - e_{22} \equiv h_{12}$, $h_2 = e_{22} - e_{33} \equiv h_{23}$, $h_3 = e_{33} - e_{44} \equiv h_{34}$, $e_1 \equiv e_{12}$, $e_2 \equiv e_{23}$, $e_3 \equiv e_{34}$, $f_1 \equiv e_{21}$, $f_2 \equiv e_{32}$ and $f_3 \equiv e_{43}$ be the standard Chevalley generators (simple roots) of $\mathcal{U}(sl(4))$. The others roots obtained by action of the Weyl group are denoted by $e_{13} = [e_{12}, e_{23}]$, $e_{14} = [e_{13}, e_{34}]$, $e_{24} = [e_{23}, e_{34}]$, $e_{31} = [e_{32}, e_{21}]$, $e_{41} = [e_{43}, e_{31}]$, $e_{42} = [e_{43}, e_{32}]$, $h_{13} = h_{12} + h_{23}$, $h_{14} = h_{12} + h_{23} + h_{34}$ and $h_{24} = h_{23} + h_{34}$. As for $\mathcal{U}_{\mathsf{h}}(sl(3))$, the Jordanian deformation arises here from the longest roots, i.e. from e_{14}, e_{41} and h_{14}. These generators are deformed as follows:

$$
\begin{aligned}
T &= \mathsf{h}e_{14} + \sqrt{1 + \mathsf{h}^2 e_{14}^2}, & T^{-1} &= -\mathsf{h}e_{14} + \sqrt{1 + \mathsf{h}^2 e_{14}^2}, \\
E_{41} &= e_{41} - \frac{\mathsf{h}^2}{4} e_{14}(h_{14}^2 - 1), & H_{14} &= \sqrt{1 + \mathsf{h}^2 e_{14}^2}\, h_{14},
\end{aligned} \tag{28}
$$

with the well-known coproducts

$$
\begin{aligned}
\Delta(T) &= T \otimes T, \qquad \Delta(T^{-1}) = T^{-1} \otimes T^{-1}, \\
\Delta(E_{41}) &= E_{41} \otimes T + T^{-1} \otimes E_{41}, \\
\Delta(H_{14}) &= H_{14} \otimes T + T^{-1} \otimes H_{14}.
\end{aligned} \tag{29}
$$

By analogy with what is happen in $\mathcal{U}_{\mathsf{h}}(sl(3))$ algebra, the subsets $\{h_{12}, e_{12}, e_{21}, e_{24}, e_{42}, h_{24} = h_{23} + h_{34}, e_{14}, e_{41}, h_{14} = h_{12} + h_{23} + h_{34}\}$ and $\{h_{13} = h_{12} + h_{23}, e_{13}, e_{31}, e_{34}, e_{43}, h_{34}, e_{14}, e_{41}, h_{14} = h_{12} + h_{23} + h_{34}\}$[6] are deformed exactly as presented above (see (10) and (11)), i.e.

$$
\begin{aligned}
E_{12} &= \sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}}\, e_{12} = T^{1/2} e_{12}, \\
E_{24} &= \sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}}\, e_{24} = T^{1/2} e_{24}, \\
E_{21} &= \sqrt{-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}}\, e_{21} + \frac{\mathsf{h}}{2}\sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}}\, e_{24} h_{14} = T^{-1/2}\Big(e_{21} + \frac{\mathsf{h}}{2} T e_{24} h_{14}\Big), \\
E_{42} &= \sqrt{-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}}\, e_{42} - \frac{\mathsf{h}}{2}\sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}}\, e_{12} h_{14} = T^{-1/2}\Big(e_{42} - \frac{\mathsf{h}}{2} T e_{12} h_{14}\Big), \\
H_{12} &= \Big(-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}\Big)\Big(\sqrt{1+\mathsf{h}^2 e_{14}^2}\, h_{12} + \frac{\mathsf{h}}{2} e_{14}(h_{12} - h_{24})\Big) = h_{12} - \frac{\mathsf{h}}{2} e_{14} T^{-1} h_{14}, \\
H_{24} &= \Big(-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}\Big)\Big(\sqrt{1+\mathsf{h}^2 e_{14}^2}\, h_{24} - \frac{\mathsf{h}}{2} e_{14}(h_{12} - h_{24})\Big) = h_{24} - \frac{\mathsf{h}}{2} e_{14} T^{-1} h_{14}
\end{aligned} \tag{30}
$$

and

$$
\begin{aligned}
E_{13} &= \sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}}\, e_{13} = T^{1/2} e_{13}, \\
E_{34} &= \sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}}\, e_{34} = T^{1/2} e_{34}, \\
E_{31} &= \sqrt{-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}}\, e_{31} + \frac{\mathsf{h}}{2}\sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}}\, e_{34} h_{14} = T^{-1/2}\Big(e_{31} + \frac{\mathsf{h}}{2} e_{34} h_{14}\Big), \\
E_{43} &= \sqrt{-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}}\, e_{43} - \frac{\mathsf{h}}{2}\sqrt{\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}}\, e_{13} h_{14} = T^{-1/2}\Big(e_{43} - \frac{\mathsf{h}}{2} e_{13} h_{14}\Big), \\
H_{13} &= \Big(-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}\Big)\Big(\sqrt{1+\mathsf{h}^2 e_{14}^2}\, h_{13} + \frac{\mathsf{h}}{2} e_{14}(h_{13} - h_{34})\Big) = h_{13} - \frac{\mathsf{h}}{2} e_{14} T^{-1} h_{14}, \\
H_{34} &= \Big(-\mathsf{h}e_{14} + \sqrt{1+\mathsf{h}^2 e_{14}^2}\Big)\Big(\sqrt{1+\mathsf{h}^2 e_{14}^2}\, h_{34} - \frac{\mathsf{h}}{2} e_{14}(h_{13} - h_{34})\Big) = h_{34} - \frac{\mathsf{h}}{2} e_{14} T^{-1} h_{14}
\end{aligned} \tag{31}
$$

The elements E_{23}, E_{32} and H_{23} are obtained after analyzing the commutators $[E_{24}, E_{43}]$ and $[E_{34}, E_{42}]$. It is simple to see that these elements remain undeformed, i.e.

$$
E_{23} = e_{23}, \qquad E_{32} = e_{32}, \qquad H_{23} = h_{23}. \tag{32}
$$

It is now easy to verify that

$$
\begin{aligned}
&H_{23} + H_{34} = H_{24}, \qquad [E_{12}, E_{23}] = E_{13}, \qquad [E_{32}, E_{21}] = E_{31}, \\
&H_{12} + H_{23} = H_{13}, \qquad [E_{23}, E_{34}] = E_{24}, \qquad [E_{43}, E_{32}] = E_{42}.
\end{aligned} \tag{33}
$$

[6]Each subsets forms a $\mathcal{U}(sl(3))$ subalgebra in $\mathcal{U}(sl(4))$.

Proposition 5 *The generating elements* $H_1 \equiv H_{12}$, $H_2 \equiv H_{23}$, $H_3 \equiv H_{34}$, $E_1 \equiv E_{12}$, $E_2 \equiv E_{23}$, $E_3 \equiv E_{34}$, $F_1 \equiv E_{21}$, $F_2 \equiv E_{32}$, $F_3 \equiv E_{43}$ *of the Jordanian quantum algebra* $\mathcal{U}_{\mathsf{h}}(sl(4))$ *obey the following commutations rules:*

$$
\begin{aligned}
&T = \Big(1 + 2\mathsf{h}[E_1,[E_2,E_3]]\Big)^{1/2}, \qquad\qquad T^{-1} = \Big(1 + 2\mathsf{h}[E_1,[E_2,E_3]]\Big)^{-1/2},\\
&[H_1,H_2] = [H_1,H_3] = [H_2,H_3] = 0,\\
&[H_1,E_1] = 2E_1, \qquad [H_1,E_2] = -E_2, \qquad [H_1,E_3] = 0,\\
&[H_2,E_1] = -E_1, \qquad [H_2,E_2] = 2E_2, \qquad [H_2,E_3] = -E_3,\\
&[H_3,E_1] = 0, \qquad [H_3,E_2] = -E_2, \qquad [H_3,E_3] = 2E_3,\\
&[H_1,F_1] = -2F_1 + T^{-1}[F_1,T](H_1+H_2+H_3), \qquad [H_1,F_2] = F_2,\\
&[H_1,F_3] = T^{-1}[F_3,T](H_1+H_2+H_3),\\
&[H_2,F_1] = F_1, \qquad [H_2,F_2] = -2F_2, \qquad [H_2,F_3] = F_3,\\
&[H_3,F_1] = T^{-1}[F_1,T](H_1+H_2+H_3), \qquad [H_3,F_2] = F_2,\\
&[H_3,F_3] = -2F_3 + T^{-1}[F_3,T](H_1+H_2+H_3),\\
&[T^{-1}E_1,F_1] = T^{-1}H_1 + \frac{1}{2}(T-T^{-1})(H_1+H_2+H_3),\\
&[E_2,F_2] = H_2,\\
&[T^{-1}E_3,F_3] = T^{-1}H_3 + \frac{1}{2}(T-T^{-1})(H_1+H_2+H_3),\\
&[T^{-1}E_1,F_2] = [T^{-1}E_1,F_3] = 0,\\
&[E_2,F_1] = [E_2,F_3] = 0,\\
&[T^{-1}E_3,F_1] = [T^{-1}E_3,F_2] = 0,\\
&[E_1,E_3] = [TF_1,TF_3] = 0,\\
&E_1^2E_2 - 2E_1E_2E_1 + E_2E_1^2 = 0, \qquad E_1E_2^2 - 2E_2E_1E_2 + E_2^2E_1 = 0,\\
&E_2^2E_3 - 2E_2E_3E_2 + E_3E_2^2 = 0, \qquad E_2E_3^2 - 2E_3E_2E_3 + E_3^2E_2 = 0,\\
&(TF_1)^2F_2 - 2TF_1F_2TF_1 + F_2(TF_1)^2 = 0, \qquad TF_1F_2^2 - 2F_2TF_1F_2 + F_2^2TF_1 = 0,\\
&(TF_3)^2F_2 - 2TF_3F_2TF_3 + F_2(TF_3)^2 = 0, \qquad F_2^2TF_3 - 2F_2TF_3F_2 + TF_3F_2^2 = 0,
\end{aligned}
\tag{34}
$$

or, briefly,

$$
\begin{aligned}
&[H_i,H_j] = 0,\\
&[H_i,E_j] = a_{ij}E_j,\\
&[H_i,F_j] = -a_{ij}F_j + (\delta_{i1}+\delta_{i3})T^{-1}[F_j,T](H_1+H_2+H_3),\\
&[T^{-(\delta_{i1}+\delta_{i3})}E_i,F_j] = \delta_{ij}\Big(T^{-(\delta_{i1}+\delta_{i3})}H_i + \frac{(\delta_{i1}+\delta_{i3})}{2}(T-T^{-1})(H_1+H_2+H_3)\Big),\\
&[E_i,E_j] = [T^{(\delta_{i1}+\delta_{i3})}F_i, T^{(\delta_{j1}+\delta_{j3})}F_j] = 0, \qquad |i-j|>1,\\
&(ad\,E_i)^{1-a_{ij}}(E_j) = 0, \qquad (i \neq j),\\
&(ad\,T^{(\delta_{i1}+\delta_{i3})}F_i)^{1-a_{ij}}(T^{(\delta_{j1}+\delta_{j3})}F_j) = 0, \qquad (i \neq j),
\end{aligned}
\tag{35}
$$

where $(a_{ij})_{i,j=1,2,3}$ *is the Cartan matrix of* $sl(4)$*.*

Proposition 6 *The non-cocommutative coproduct structure of* $\mathcal{U}_{\mathsf{h}}(sl(4))$ *reads:*

$$\begin{aligned}
&\Delta(E_1) = E_1 \otimes 1 + T \otimes E_1,\\
&\Delta(E_2) = E_2 \otimes 1 + 1 \otimes E_2,\\
&\Delta(E_3) = E_3 \otimes 1 + T \otimes E_3,\\
&\Delta(F_1) = F_1 \otimes 1 + T^{-1} \otimes F_1 + (H_1 + H_2 + H_3) \otimes T^{-1}[F_1, T],\\
&\Delta(F_2) = F_2 \otimes 1 + T^{-1} \otimes F_2,\\
&\Delta(F_3) = F_3 \otimes 1 + T^{-1} \otimes F_3 + (H_1 + H_2 + H_3) \otimes T^{-1}[F_3, T],\\
&\Delta(H_1) = H_1 \otimes 1 + 1 \otimes H_1 - \frac{1}{2}(1 - T^{-2}) \otimes (H_1 + H_2 + H_3),\\
&\Delta(H_2) = H_2 \otimes 1 + 1 \otimes H_2,\\
&\Delta(H_3) = H_3 \otimes 1 + 1 \otimes H_3 - \frac{1}{2}(1 - T^{-2}) \otimes (H_1 + H_2 + H_3).
\end{aligned} \tag{36}$$

In the (fund.) $\otimes$ (arb.) representation, the $R_{\mathsf{h}} = (\pi_{(fund.)} \otimes \pi_{(arb.)})\mathcal{R}_{\mathsf{h}}$ take the following simple form:

$$R_{\mathsf{h}} = \begin{pmatrix} T & 2\mathsf{h}T^{-1/2}e_{24} & 2\mathsf{h}T^{-1/2}e_{34} & -\frac{\mathsf{h}}{2}(T+T^{-1})(h_1+h_2+h_3) + \frac{\mathsf{h}}{2}(T-T^{-1}) \\ 0 & I & 0 & -2\mathsf{h}T^{1/2}e_{12} \\ 0 & 0 & I & -2\mathsf{h}T^{1/2}e_{13} \\ 0 & 0 & 0 & T^{-1} \end{pmatrix}. \tag{37}$$

Proposition 7 *The universal* $\mathcal{R}_{\mathsf{h}}$*-matrix for* $\mathcal{U}_{\mathsf{h}}(sl(4))$ *may be cast in the form:*

$$\mathcal{R}_{\mathsf{h}} = \mathcal{F}_{21}^{-1}\mathcal{F}, \tag{38}$$

where

$$\mathcal{F} = \exp\Big(\mathsf{h}TH_{14} \otimes E_{14}\Big) \exp\Big(2\mathsf{h}TE_{34} \otimes T^{-2}E_{13} + 2\mathsf{h}TE_{24} \otimes T^{-2}E_{12}\Big), \tag{39}$$

$$E_{14} = \mathsf{h}^{-1}\ln T = \mathsf{h}^{-1}\,\mathrm{arcsinh}\,\mathsf{h}e_{14}. \tag{40}$$

The $\mathcal{R}_{\mathsf{h}}$-matrix (38) coincides with the universal $\mathcal{R}$-matrix of the Borel subalgebra. Let us just note that the tensor elements $TE_{34} \otimes T^{-2}E_{13}$ and $TE_{24} \otimes T^{-2}E_{12}$ commute.

4 $\mathcal{U}_{\mathsf{h}}(sl(N))$: Generalization

The $\mathcal{U}_{\mathsf{h}}(sl(5))$ algebra is derived in a similar way: The elements E_2, E_3, F_2, F_3, H_2, H_3 are not affected by the nonstandard quantization. From these above studies, It is easy to see that:

Proposition 8 *The Jordanian quantization deform $\mathcal{U}_{\mathsf{h}}(sl(N))$'s Chevalley generators as follows:*

$$
\begin{aligned}
T &= \mathsf{h}[e_1,[e_2,\cdots,[e_{N-2},e_{N-1}]\cdots]] + \sqrt{1+\mathsf{h}^2([e_1,[e_2,\cdots,[e_{N-2},e_{N-1}]\cdots]])^2},\\
T^{-1} &= -\mathsf{h}[e_1,[e_2,\cdots,[e_{N-2},e_{N-1}]\cdots]] + \sqrt{1+\mathsf{h}^2([e_1,[e_2,\cdots,[e_{N-2},e_{N-1}]\cdots]])^2},\\
E_i &= T^{(\delta_{i1}+\delta_{i,N-1})/2} e_i,\\
F_i &= T^{-(\delta_{i1}+\delta_{i,N-1})/2}\Big(f_i + \frac{\mathsf{h}}{2}T[f_i,[e_1,[e_2,\cdots,[e_{N-2},e_{N-1}]\cdots]]](h_1+\cdots+h_{N-1})\Big)\\
H_i &= h_i - \frac{(\delta_{i1}+\delta_{i,N-1})\mathsf{h}}{2}[e_1,[e_2,\cdots,[e_{N-2},e_{N-1}]\cdots]]T^{-1}(h_1+\cdots+h_{N-1})
\end{aligned}
\tag{41}
$$

($i=1,\cdots,\ N-1$) and they satisfy the commutation relations

$$
\begin{aligned}
&[H_i,H_j]=0,\\
&[H_i,E_j]=a_{ij}E_j,\\
&[H_i,F_j]=-a_{ij}F_j+(\delta_{i1}+\delta_{i,N-1})T^{-1}[F_j,T](H_1+\cdots+H_{N-1}),\\
&[T^{-(\delta_{i1}+\delta_{i,N-1})}E_i,F_j]=\delta_{ij}\Big(T^{-(\delta_{i1}+\delta_{i,N-1})}H_i+\frac{(\delta_{i1}+\delta_{i,N-1})}{2}(T-T^{-1})(H_1+\cdots+H_{N-1})\Big),\\
&[E_i,E_j]=0, \qquad |i-j|>1,\\
&[T^{(\delta_{i1}+\delta_{i,N-1})}F_i,T^{(\delta_{j1}+\delta_{j,N-1})}F_j]=0, \qquad |i-j|>1,\\
&(ad\,E_i)^{1-a_{ij}}(E_j)=0, \qquad (i\neq j),\\
&(ad\,T^{(\delta_{i1}+\delta_{i,N-1})}F_i)^{1-a_{ij}}(T^{(\delta_{j1}+\delta_{j,N-1})}F_j)=0, \qquad (i\neq j),
\end{aligned}
\tag{42}
$$

where $(a_{ij})_{i,j=1,\cdots,N}$ is the Cartan matrix of $sl(N)$, i.e. $a_{ii}=2$, $a_{i,i\pm1}=-1$ and $a_{ij}=0$ for $|i-j|>1$.

The algebra (42) is called the *Jordanian quantum algebra* $\mathcal{U}_{\mathsf{h}}(sl(N))$. The expressions (41) may be regarded as a particular nonlinear realization of the $\mathcal{U}_{\mathsf{h}}(sl(N))$ generators.

Proposition 9 *The Jordanian algebra $\mathcal{U}_{\mathsf{h}}(sl(N))$ (42) admits the following coalgebra structure:*

$$
\begin{aligned}
\Delta(E_i) &= E_i\otimes 1 + T^{(\delta_{i1}+\delta_{i,N-1})}\otimes E_i,\\
\Delta(F_i) &= F_i\otimes 1 + T^{-(\delta_{i1}+\delta_{i,N-1})}\otimes F_i + T(H_1+\cdots+H_{N-1})\otimes T^{-1}[F_i,T],\\
\Delta(H_i) &= H_i\otimes 1 + 1\otimes H_i - \frac{(\delta_{i1}+\delta_{i,N-1})}{2}(1-T^{-2})\otimes(H_1+\cdots+H_{N-1}),\\
S(E_i) &= -T^{-(\delta_{i1}+\delta_{i,N-1})}E_i,\\
S(F_i) &= -T^{(\delta_{i1}+\delta_{i,N-1})}F_i + T^2(H_1+\cdots+H_{N-1})T^{-2}[F_i,T],\\
S(H_i) &= -H_i + \frac{(\delta_{i1}+\delta_{i,N-1})}{2}(1-T^2)(H_1+\cdots+H_{N-1}),\\
\epsilon(E_i) &= \epsilon(F_i)=\epsilon(H_i)=0.
\end{aligned}
\tag{43}
$$

Proposition 10 *The $\mathcal{R}_{\mathsf{h}}$-matrix of $\mathcal{U}_{\mathsf{h}}(sl(N))$ has the following general form:*

$$\mathcal{R}_{\mathsf{h}} = \mathcal{F}_{21}^{-1}\mathcal{F}, \tag{44}$$

where

$$\mathcal{F} = \exp\Big(\mathsf{h}TH_{1N} \otimes E_{1N}\Big) \exp\Big(\sum_{k=2}^{N-1} 2\mathsf{h}TE_{kN} \otimes T^{-2}E_{1k}\Big), \tag{45}$$

$$H_{1N} = T(H_1 + \cdots H_{N-1}), \tag{46}$$

$$E_{1N} = \mathsf{h}^{-1}\ln T = \mathsf{h}^{-1}\,\mathrm{arcsinh}\,\mathsf{h}e_{1N}, \tag{47}$$

$$E_{kN} = [E_k, [\cdots, [E_{N-2}, E_{N-1}]]], \qquad k = 2, \cdots, N-2, \tag{48}$$

$$E_{N-1,N} = E_{N-1}, \tag{49}$$

$$E_{12} = E_1, \tag{50}$$

$$E_{1k} = [E_1, [\cdots, [E_{k-2}, E_{k-1}]]], \qquad k = 3, \cdots, N-1 \tag{51}$$

and may be obtained from the $\mathcal{R}_q$-matrix associated to $\mathcal{U}_q(sl(N))$ via the contraction procedure discussed above, i.e.

$$\mathcal{R}_{\mathsf{h}} = \lim_{q\to 1}\Big[E_q\Big(\frac{\mathsf{h}\hat{e}_{1N}}{q-1}\Big) \otimes E_q\Big(\frac{\mathsf{h}\hat{e}_{1N}}{q-1}\Big)\Big]^{-1}\mathcal{R}_q\Big[E_q\Big(\frac{\mathsf{h}\hat{e}_{1N}}{q-1}\Big) \otimes E_q\Big(\frac{\mathsf{h}\hat{e}_{1N}}{q-1}\Big)\Big]. \tag{52}$$

It is interesting to note that, via the nonlinear map (41), the h-deformed generators (E_i, F_i, H_i) may be also equipped with an induced co-commutative coproduct. Similarly, the undeformed generators (e_i, f_i, h_i), via the inverse map, may be viewed as elements of the $\mathcal{U}_{\mathsf{h}}(sl(N))$ algebra; and, thus, may be endowed with an induced noncommutative coproduct.

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